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Global existence of solutions for a fourth-order nonlinear Schrödinger equation[☆]

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Abstract

In this work we study the Cauchy problem of a fourth-order nonlinear Schrödinger equation which arises from certain physical applications. We consider only the cases $n = 1, 2, 3$. Local existence of solutions for initial data belonging to Sobolev spaces with index greater than $n/2$ is established by using the standard contraction mapping argument. The main topic is proving that the solution is global if either the exponent of the nonlinear term is sub-critical or it is critical or super-critical but the initial data are small. This result extends the corresponding result of Fibich et al. obtained in 2002 to the super-critical case and to a more general equation. The analysis is based on applications of conservation laws for this equation.

Keywords: Nonlinear Schrödinger equation; Fourth order; Initial value problem; Global existence

1. Introduction

In this work we study the Cauchy problem of the following fourth-order nonlinear Schrödinger equation:

$$i\varphi_t + \lambda\Delta\varphi + \mu\Delta^2\varphi + f(|\varphi|^2)\varphi = 0, \qquad x \in R^n, t \in R, \tag{1.1}$$

where λ and μ are real numbers, $\mu \neq 0$, and f is a given real-valued nonlinear function.

When $\mu = 0$ and $\lambda \neq 0$, Eq. (1.1) is the nonlinear Schrödinger equation which arises in many scientific fields such as quantum mechanics, nonlinear optics, and plasma physics, and has been intensively studied by many authors. When $\lambda = 0$, $\mu \neq 0$ and $f(s) = s^\sigma$ $(\sigma \geq 1)$, Eq. (1.1) reduces to the *biharmonic* nonlinear Schrödinger equation

$$i\varphi_t + \mu\Delta^2\varphi + |\varphi|^{2\sigma}\varphi = 0, \tag{1.2}$$

which was studied by Ivanov and Kosevich [1] and Turitsyn [2] in the context of the stability of solitons in magnetic materials where the effective quasi-particle mass becomes infinite. The case where both the second-order dispersion

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term $\lambda\Delta\varphi(\lambda \neq 0)$ and the fourth-order dispersion term $\mu\Delta^2\varphi(\mu \neq 0)$ exist was considered by Karpman and Shagalov ([3] and the references therein) and Fibich et al. [4], who studied the equation

$$i\varphi_t + \lambda\Delta\varphi + \mu\Delta^2\varphi + \nu|\varphi|^{2\sigma}\varphi = 0 \tag{1.3}$$

with $\lambda = 1$, $\mu \neq 0$, and $\nu = 1$. Karpman and Shagalov studied the so-called waveguide solutions and their stability, while Fibich et al. studied solutions of the general Cauchy problem. The main results of Fibich et al. [4] can be summarized as follows: Let $\lambda = 1$ and $\nu = 1$. Then each of the following three conditions is sufficient for global existence of solutions of the Cauchy problem of (1.3): (i) $\mu > 0$; (ii) $\mu < 0$ and $\sigma n < 4$; (iii) $\mu < 0$, $\sigma n = 4$, and $\|\varphi_0\|^2_{H^2(R^n)} < N_c^B$, where φ_0 is the initial value of φ, and N_c^B is a positive number depending only on the dimension n. The case $\sigma n > 4$ is open.

In this work we study the global existence of solutions for the Cauchy problem of the more general nonlinear Schrödinger equation (1.1). We want to extend the results of [4] stated above. In particular, we want to establish global existence for the case $\sigma n > 4$ for small initial value problems. Later on we shall use the notation φ_0 to denote the initial value of the unknown function φ, i.e., $\varphi_0(x) = \varphi(x, 0)$ for all $x \in R^n$.

The main results of this work are as follows:

Theorem 1.1 (*For General Initial Value*). *Let $n = 1, 2, 3$ and $\mu \neq 0$. Let k be an integer, $k \geq 2$. Suppose $f \in C^{k+1}(R_+; R)$, and there exist a positive constant c_0 and an exponent $0 \leq \sigma < \frac{4}{n}$ such that*

$$\mu f(s) \geq -c_0(1+s)^\sigma \qquad \text{for } s \geq 0. \tag{1.4}$$

Then for any $\varphi_0 \in H^k(R^n)$, the initial value problem of (1.1) has a unique global solution φ in the class

$$C(R; H^k(R^n)) \bigcap C^1(R; H^{k-2}(R^n)). \tag{1.5}$$

Theorem 1.2 (*For Small Initial Value*). *Let $n = 1, 2, 3$ and $\mu \neq 0$. Let k be an integer, $k \geq 2$. Suppose $f \in C^{k+1}(R_+; R)$. Then there exists a constant $c^* > 0$ such that for any $\varphi_0 \in H^k(R^n)$ satisfying $\|\varphi_0\|_{2,2} \leq c^*$, the initial value problem of (1.1) has a unique global solution φ in the class (1.5).*

Applying these results to the Eq. (1.3) we immediately get the following result:

Theorem 1.3. *Let $n = 1, 2, 3$ and $\mu \neq 0$. Let σ be a nonnegative integer. Assume that $\varphi_0 \in H^k(R^n)$ for some integer $k \geq 2$. Then each of the following three conditions is sufficient for global existence of solutions to the initial value problem of the Eq. (1.3) in the class (1.5) :*

(1) $\mu\nu > 0$;
(2) $\mu\nu < 0$, *and* $0 \leq \sigma < \frac{4}{n}$;
(3) $\mu\nu < 0$, $\sigma \geq \frac{4}{n}$, *and* $\|\varphi_0\|_{2,2} \leq c^*$, *where c^* is a positive constant (see Lemma 3.3).*

Remark 1. The condition that σ is a nonnegative integer in the last result is imposed to ensure that the function $s \to s^\sigma$ (for $s \geq 0$) is sufficiently smooth at $s = 0$. This condition can be removed by using an approximation argument. However, due to limitations of space, we shall not do this job.

Remark 2. We conjecture that, like in the nonlinear Schrödinger equation case, there is an optimal constant for the critical value c^* appearing in Theorems 1.2 and 1.3, and this optimal constant should have relations with the ground state of the fourth-order Schrödinger equation and some minimizer of certain Sobolev inequalities (cf. [4]). To prove this conjecture needs much more work; we leave it for the future.

In Section 2 we establish local existence. In Section 3 we give the proofs of Theorems 1.1 and 1.2.

2. Local existence of solutions

Lemma 2.1. *Let k be an integer greater than $\frac{n}{2}$. Let f be a real-valued function defined on $R_+ = [0, \infty)$. Suppose $f \in C^k(R_+; R)$. Then for any $u \in H^k(R^n)$ we have $f(|u|^2)u \in H^k(R^n)$, and there exists $\omega_1 \in C(R_+; R_+)$ such that for any $u \in H^k(R^n)$,*

$$\|f(|u|^2)u\|_{k,2} \leq \omega_1(\|u\|_{k,2})\|u\|_{k,2}. \tag{2.1}$$

Suppose further $f \in C^{k+1}(R_+; R)$. *Then there exists* $\omega_2 \in C(R_+; R_+)$ *such that for any* $u_1, u_2 \in H^k(R^n)$,

$$\| f(|u_1|^2)u_1 - f(|u_2|^2)u_2\|_{k,2} \le \omega_2(\|u_1\|_{k,2} + \|u_2\|_{k,2})\|u_1 - u_2\|_{k,2}, \tag{2.2}$$

and there exists $\omega_3 \in C(R_+; R_+)$ *such that for any* $u \in H^{k+1}(R^n)$,

$$\| f(|u|^2)u\|_{k+1,2} \le \omega_1(\|u\|_{k,2})\|u\|_{k+1,2}. \tag{2.3}$$

The proof can be found in the book of Mizohata [5], p. 355.

As is well known, the initial value problem of Eq. (1.1) is equivalent to the following integral equation:

$$\varphi(\cdot, t) = W(t)\varphi_0 + i\int_0^t W(t-s) f(|\varphi(\cdot, s)|^2)\varphi(\cdot, s)\mathrm{d}s, \tag{2.4}$$

where $W(t)(t \in R)$ is the operator on $S'(R^n)$ defined by

$$W(t)u = F^{-1}(\mathrm{e}^{i(\mu|\xi|^4 t - \lambda|\xi|^2 t)}F(u)) \qquad \text{for } u \in S'(R^n), \tag{2.5}$$

where F and F^{-1} denote the Fourier and inverse Fourier transformations over R^n, respectively. By the Plancherel identity, it is clear that for any $t \in R$ and any real s, $W(t)$ is a unitary operator when restricted to $H^s(R^n)$, i.e., the following identity holds:

$$\|W(t)u\|_{s,2} = \|u\|_2 \qquad \text{for } u \in H^s(R^n). \tag{2.6}$$

Theorem 2.2. *Let* k *be an integer greater than* $\frac{n}{2}$. *Suppose* $f \in C^{k+1}(R_+; R)$. *Then for any* $\varphi_0 \in H^k(R^n)$, *there exists* $T > 0$ *such that the initial value problem of Eq.* (1.1) *has a unique solution* $\varphi = \varphi(x, t)$ *on* $R^n \times [-T, T]$, *belonging to the class*

$$C([-T, T]; H^k(R^n)) \bigcap C^1([-T, T]; H^{k-4}(R^n)). \tag{2.7}$$

Furthermore, if we denote by T^* *the supremum of all* $T > 0$ *such that the above assertion holds, then either* $T^* = \infty$ *or* $T^* < \infty$ *and*

$$\begin{aligned} &\text{either} \quad \limsup_{t \to T^*-0} \|\varphi(\cdot, t)\|_{[\frac{n}{2}]+1,2} = \infty, \\ &\text{or} \quad \limsup_{t \to -T^*+0} \|\varphi(\cdot, t)\|_{[\frac{n}{2}]+1,2} = \infty. \end{aligned} \tag{2.8}$$

Proof. Without loss of generality we may assume that $\varphi_0 \neq 0$; if $\varphi_0 = 0$ then clearly $\varphi = 0$ solves the problem. For given integer $k > \frac{n}{2}$ and real $T > 0$, we denote by X_T the Banach space $C([-T, T]; H^k(R^n))$ (with norm $|||\varphi||| = \sup_{|t|\le T} \|\varphi(\cdot, t)\|_{k,2}$). Let $\mathcal{F}$: $X_T \to X_T$ be the operator defined by

$$(\mathcal{F}\varphi)(\cdot, t) = W(t)\varphi_0 + i\int_0^t W(t-s) f(|\varphi(\cdot, s)|^2)\varphi(\cdot, s)\mathrm{d}s.$$

By (2.1) and (2.6), it is clear that $\mathcal{F}$ is well defined. Actually, using (2.1), (2.2) and (2.6) and a standard argument, we can easily prove that $\mathcal{F}$ maps the closed ball in X_T

$$B_{M,T} = \{\varphi \in X_T : \ |||\varphi||| \le M\}, \qquad M = 2\|\varphi_0\|_{k,2},$$

into itself and is a contraction mapping on this ball provided T is sufficiently small (say, $T < \min\{(2M\omega_1(M))^{-1}, (2\omega_2(2M))^{-1}\}$). Hence, by the Banach fixed point theorem we see that $\mathcal{F}$ has a unique fixed point on $B_{M,T}$ for small T, which is clearly the unique solution of the initial value problem of (1.1) in the class $C([-T, T]; H^k(R^n))$. From (1.1) we see that also $\varphi \in C^1([-T, T]; H^{k-4}(R^n))$.

In the remainder we assume that $T^* < \infty$ and prove (2.8). Consider first the case $k = [\frac{n}{2}] + 1$. If (2.8) does not hold then there is $M > 0$ such that

$$\|\varphi(\cdot, t)\|_{k,2} \le M \qquad \text{for } -T^* < t < T^*.$$

By an argument similar to that used before, we can deduce that there exists a constant $\delta > 0$ depending only on M, such that for each $t_0 \in (-T^*, T^*)$ Eq. (1.1) has a unique solution on $R^n \times [t_0 - \delta, t_0 + \delta]$ such that its value at t_0 is $\varphi(\cdot, t_0)$ and it belongs to

$$C([t_0 - \delta, t_0 + \delta]; H^k(R^n)) \bigcap C^1([t_0 - \delta, t_0 + \delta]; H^{k-4}(R^n)).$$

By uniqueness, all these solutions are mutually equal on their common domains. It follows that we can paste these solutions together to get a solution on the domain $R^n \times (-T^* - \delta, T^* + \delta)$, such that for any $0 < T < T^* + \delta$ it belongs to the class (2.7), which contradicts the definition of T^*. Consider next the case $k > [\frac{n}{2}]+1$. Let $k_0 = [\frac{n}{2}]+1$. If (2.8) does not hold then there is $M_0 > 0$ such that

$$\|\varphi(\cdot, t)\|_{k_0,2} \le M_0 \qquad \text{for } -T^* < t < T^*.$$

By (2.3) and (2.6) we have, for any $t \in (-T^*, T^*)$,

$$\begin{aligned}\|\varphi(\cdot, t)\|_{k_0+1,2} &\le \|\varphi_0\|_{k_0+1,2} + \left|\int_0^t \omega_3(\|\varphi(\cdot, s)\|_{k_0,2})\|\varphi(\cdot, s)\|_{k_0+1,2}\mathrm{d}s\right| \\ &\le \|\varphi_0\|_{k_0+1,2} + \omega_3(M_0)\left|\int_0^t \|\varphi(\cdot, s)\|_{k_0+1,2}\mathrm{d}s\right|.\end{aligned}$$

It follows from the Gronwall inequality that

$$\|\varphi(\cdot, t)\|_{k_0+1,2} \le \|\varphi_0\|_{k_0+1,2}\mathrm{e}^{\omega_3(M_0)|t|} \le \|\varphi_0\|_{k_0+1,2}\mathrm{e}^{\omega_3(M_0)T^*} \equiv M_1$$

for $t \in (-T^*, T^*)$. Repeating this argument we finally get, by induction, that

$$\|\varphi(\cdot, t)\|_{k,2} \le \|\varphi_0\|_{k,2}\mathrm{e}^{\omega_3(M_{p-1})T^*} \equiv M_p$$

for $t \in (-T^*, T^*)(p = k - k_0)$. Hence $\|\varphi(\cdot, t)\|_{k,2}$ is bounded for $t \in (-T^*, T^*)$. Using an argument similar to that for the case $k = k_0$, we can then deduce that the solution can be extended to a larger domain $R^n \times (-T^* - \delta, T^* + \delta)$ for some $\delta > 0$, such that it belongs to the class (2.7) for any $0 < T < T^* + \delta$, which contradicts the definition of T^*. This completes the proof. □

By the second assertion of Theorem 2.2, it follows that if for any $T > 0$ we can get an a priori estimate $\sup_{|t|<T} \|\varphi(\cdot, t)\|_{k_0,2} \le C(T)(k_0 = [\frac{n}{2}] + 1)$ for the solution of Eq. (1.1), then the solution exists for all $(x, t) \in R^n \times R$.

3. Global existence of solutions

Lemma 3.1. *Let $F(s) = \int_0^s f(s')\mathrm{d}s'$ and let φ be the solution of (1.1) on $R^n \times (-T, T)$. Assume that f is continuous and for some integer k, $k \ge 2$ and $k > \frac{n}{2}$,*

$$\varphi \in C((-T, T); H^k(R^n)) \bigcap C^1((-T, T); H^{k-4}(R^n)).$$

Then for any $|t| < T$,

$$\|\varphi(\cdot, t)\|_2^2 = \|\varphi_0\|_2^2 \equiv E_0, \tag{3.1}$$

$$\begin{aligned}&\|\Delta\varphi(\cdot, t)\|_2^2 - \frac{\lambda}{\mu}\|\nabla\varphi(\cdot, t)\|_2^2 + \frac{1}{\mu}\int_{R^n} F(|\varphi(\cdot, s)|^2)\mathrm{d}x \\ &\quad = \|\Delta\varphi_0\|_2^2 - \frac{\lambda}{\mu}\|\nabla\varphi_0\|_2^2 + \frac{1}{\mu}\int_{R^n} F(|\varphi_0|^2)\mathrm{d}x \equiv E_1.\end{aligned} \tag{3.2}$$

Proof. We first note that $\varphi \in C((-T, T); (C \cap L^\infty)(R^n))$, by the assumption on k and φ. Thus all terms in (3.2) make sense. Multiplying (1.1) by $\overline{\varphi}$, integrating over R^n and then taking imaginary parts we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{R^n} |\varphi(x, t)|^2\mathrm{d}x = 0.$$

Hence (3.1) holds. Next, multiplying (1.1) by $\overline{\varphi_t}$, integrating over R^n and taking real parts, we obtain

$$-\lambda \frac{\mathrm{d}}{\mathrm{d}t}\int_{R^n} |\nabla\varphi(x,t)|^2 \mathrm{d}x + \mu \frac{\mathrm{d}}{\mathrm{d}t}\int_{R^n} |\Delta\varphi(x,t)|^2 \mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_{R^n} F(|\varphi(x,t)|^2)\mathrm{d}x = 0.$$

Hence we have (3.2). □

In the remainder we always assume that $n = 1, 2, 3$. Note that under this assumption the condition $k \geq 2$ implies that $k > \frac{n}{2}$.

Lemma 3.2. *Suppose the conditions of Theorem 1.1 are satisfied. Then for any $\varphi_0 \in H^k(R^n)$ we have, for t in any existence interval of the solution φ,*

$$\|\varphi(\cdot,t)\|_{2,2} \leq \omega(\|\varphi_0\|_{2,2}), \tag{3.3}$$

where $\omega(\|\varphi_0\|_{2,2})$ represents a constant depending only on $\|\varphi_0\|_{2,2}$.

Proof. By the Agmon–Douglas–Nirenberg inequality we have

$$\|\varphi(\cdot,t)\|_{2,2} \leq C(\|\Delta\varphi(\cdot,t)\|_2 + \|\varphi(\cdot,t)\|_2).$$

It follows from (3.2) and (3.1) that

$$\|\varphi(\cdot,t)\|_{2,2} \leq -\frac{C}{\mu}\int_{R^n} F(|\varphi(x,t)|^2)\mathrm{d}x + C\|\nabla\varphi(\cdot,t)\|_2^2 + CE_0 + CE_1. \tag{3.4}$$

By the Gargliardo–Nirenberg inequality and the Cauchy inequality we have, for any $\varepsilon > 0$,

$$\|\nabla\varphi\|_2 \leq C\|\varphi\|_{2,2}^{\frac{1}{2}}\|\varphi\|_2^{\frac{1}{2}} \leq \varepsilon\|\varphi\|_{2,2} + C(\varepsilon)\|\varphi\|_2.$$

Substituting this estimate into (3.4), using (3.1) and taking ε sufficiently small, we obtain

$$\|\varphi(\cdot,t)\|_{2,2}^2 \leq -\frac{C}{\mu}\int_{R^n} F(|\varphi(x,t)|^2)\mathrm{d}x + CE_0 + CE_1.$$

By (1.4) we have $-\mu^{-1}F(s^2) \leq Cs^{2(\sigma+1)} + Cs^2$. Hence

$$\|\varphi(\cdot,t)\|_{2,2}^2 \leq C\|\varphi(\cdot,t)\|_{2(\sigma+1)}^{2(\sigma+1)} + CE_0 + CE_1. \tag{3.5}$$

Since $\sigma < \frac{4}{n}$, by the Gargliardo–Nirenberg inequality and the Young inequality we have, for any $\varepsilon > 0$,

$$\begin{aligned}\|\varphi\|_{2(\sigma+1)}^{2(\sigma+1)} &\leq C\|\varphi\|_{2,2}^{\frac{n\sigma}{2}}\|\varphi\|_2^{2(\sigma+1)-\frac{n\sigma}{2}} \\ &\leq \varepsilon\|\varphi\|_{2,2}^2 + C(\varepsilon)\|\varphi\|_2^{2+8\sigma/(4-n\sigma)} \leq \varepsilon\|\varphi\|_{2,2}^2 + C(\varepsilon)E_0^{1+4\sigma/(4-n\sigma)}.\end{aligned} \tag{3.6}$$

Substituting (3.6) into (3.5) and taking ε sufficiently small, we obtain

$$\|\varphi(\cdot,t)\|_{2,2}^2 \leq CE_0^{1+4\sigma/(4-n\sigma)} + CE_0 + CE_1 \leq \omega(\|\varphi_0\|_{2,2}).$$

Hence (3.3) holds. □

By Lemma 3.2 and Theorem 2.2, we see that Theorem 1.1 follows.

Lemma 3.3. *Suppose the conditions of Theorem 1.2 are satisfied. Then there exist constants $c^* > 0$ and $C > 0$ such that for any $\varphi_0 \in H^k(R^n)$ satisfying $\|\varphi_0\|_{2,2} \leq c^*$, we have $\|\varphi(\cdot,t)\|_{2,2} \leq C$ for any t in the existence interval of φ.*

Proof. We split the proof into four steps.

*Step*1: Clearly $f(s)$ is a continuous function, so there is a constant $c_0 > 0$ such that

$$|F(s)| \leq c_0 s \qquad \text{for } |s| \leq 1. \tag{3.7}$$

We first prove that there exists a constant $C > 0$ independent of φ_0 and T such that if

$$\|\varphi(\cdot,t)\|_\infty \leq 1 \qquad \text{for } |t| < T, \tag{3.8}$$

then

$$\|\varphi(\cdot,t)\|_{2,2} \le C\|\varphi_0\|_{2,2} \qquad \text{for } |t| < T. \tag{3.9}$$

Indeed, by an argument similar to that in the proof of Lemma 3.2, we get from (3.2) and (3.1) that

$$\|\varphi(\cdot,t)\|_{2,2}^2 \le C\int_{R^n} |F(|\varphi(x,t)|^2)|\mathrm{d}x + CE_0 + CE_1,$$

where E_0 and E_1 are as in (3.1) and (3.2). Thus if (3.8) holds then by (3.7) and (3.1) we have

$$\|\varphi(\cdot,t)\|_{2,2}^2 \le C\int_{R^n} |\varphi(x,t)|^2\mathrm{d}x + CE_0 + CE_1 \le CE_0 + CE_1.$$

Clearly $E_0 \le \|\varphi_0\|_{2,2}^2$. Since (3.8) implies that $\|\varphi_0\|_\infty \le 1$, we have also $E_1 \le C\|\varphi_0\|_{2,2}^2$. Hence (3.9) follows.

Step 2: We prove that for any $M > 0$ there exists a corresponding constant $\varepsilon > 0$ such that if $\|\varphi_0\|_2 \le \varepsilon$ and $\|\varphi(\cdot,t)\|_{2,2} \le M$ for $|t| < T$, then (3.9) holds.

Indeed, by the Gargliardo–Nirenberg inequality and (3.1) we have

$$\|\varphi(\cdot,t)\|_\infty \le C\|\varphi(\cdot,t)\|_2^{1-\frac{n}{4}}\|\varphi(\cdot,t)\|_{2,2}^{\frac{n}{4}} \le C\|\varphi_0\|_2^{1-\frac{n}{4}}\|\varphi(\cdot,t)\|_{2,2}^{\frac{n}{4}} \le CM^{\frac{n}{4}}\|\varphi_0\|_2^{1-\frac{n}{4}}.$$

It follows that if $CM^{\frac{n}{4}}\|\varphi_0\|_2^{1-\frac{n}{4}} \le 1$, then (3.8) holds, and the desired assertion follows from the assertion obtained in Step 1.

Step 3: We prove that for any given $M_0 > 0$ there exist corresponding constants $\varepsilon > 0$ such that if $\|\varphi_0\|_{2,2} \le M_0$ and $\|\varphi_0\|_2 \le \varepsilon$, then for any $T > 0$ we have

$$\|\varphi(\cdot,t)\|_{2,2} \le CM_0 \qquad \text{for } |t| < T. \tag{3.10}$$

To prove this assertion we write $M = CM_0 + 1$, where C is the constant in (3.9). Let ε be the small constant obtained in Step 2 when M is given in this way. It follows from the result of Step 2 that if $\|\varphi_0\|_2 \le \varepsilon$ and $\|\varphi(\cdot,t)\|_{2,2} \le M$, for $|t| < T_0$ for some $0 < T_0 \le T$, then $\|\varphi(\cdot,t)\|_{2,2} \le CM_0$ for $|t| < T_0$. Since clearly $C \ge 1$, we have $M > M_0$. Hence, by continuity, the condition $\|\varphi_0\|_{2,2} \le M_0$ implies that there exists $\delta > 0$ such that $\|\varphi(\cdot,t)\|_{2,2} \le M$ for all $|t| \le \delta$. Denoting by T^* the supremum of all such δ, we can conclude that $T^* = T$. Indeed, since $\|\varphi(\cdot,t)\|_{2,2} \le M$ for all $|t| < T^*$, we have $\|\varphi(\cdot,t)\|_{2,2} \le CM_0$ for $|t| < T^*$. If $T^* < T$ then by continuity we must have $\|\varphi(\cdot,t)\|_{2,2} \le CM_0 < M$ for $|t| = T^*$. It follows again by continuity that $\|\varphi(\cdot,t)\|_{2,2} < M$ for $|t| < T^* + \delta$, for some $\delta > 0$, which contradicts the definition of T^*. Hence $T^* = T$, and (3.10) follows.

Step 4: We now take $M_0 = 1$, and let ε be the small number obtained in Step 3. Then, by taking $c^* = \min\{1, \varepsilon\}$, we readily see that the desired assertion follows. This completes the proof. □

By Lemma 3.3 and Theorem 2.2, we see that Theorem 1.2 follows.

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